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Hydromagnetic transverse instability of two highly viscous fluid-particle flows with finite ion Larmor radius corrections

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Abstract. A linear analysis of the combined effect of viscosity, finite ion Larmor radius and suspended particles on Kelvin-Helmholtz instability of two superposed incompressible fluids in the presence of a uniform magnetic field is carried out. The magnetic field is assumed to be transverse to the direction of streaming. A general dispersion relation for such a configuration has been obtained using appropriate boundary conditions. The stability analysis is discussed analytically, and the obtained results are numerically confirmed. Some special cases are recovered and corrected. The limiting cases of absence of suspended particles (or fluid velocities) and finite Larmor radius, absence of suspended particles are discussed in detail. In both cases, all other physical parameters are found to have stabilizing as well as destabilizing effects on the considered system. In the former case, the kinematic viscosity is found to has a stabilizing effect, while in the later case, the finite Larmor radius is found to has a stabilizing influence for a vortex sheet. It is shown also that both finite Larmor radius and kinematic viscosity stabilizations for interchange perturbations are similar to the stabilization effect due to a magnetic field for non-interchange perturbations.

PACS. 47.20.-k Hydrodynamic stability – 47.20.Gv Viscous instability – 47.15.Pn Laminar suspensions

1 Introduction

Kelvin-Helmholtz instability arises when the layers of fluids slipping past each other have relative motion. Most of the authors have studied the hydromagnetic Kelvin-Helmholtz instability in plasmas with a view to analyze the instability of the interface or the tangential discontinuity between the two relatively moving plasma layers. Chandrasekhar [1] has given a comprehensive survey of the hydromagnetic version of this instability. He has treated the problem considering a uniform magnetic field along and transverse to the direction of streaming, and found that the magnetic field in the direction of streaming has a stabilizing influence of the Kelvin-Helmholtz instability; whereas it does not affect the stability in the transverse direction. An excellent review of the Kelvin-Helmholtz has been presented by Gerwin [2]. It is well-known that in the absence of a magnetic field, a velocity discontinuity at a plane interface between two inviscid, incompressible fluids is always unstable for any value, however, small of the relative velocity. If, however, the fluids are perfectly conducting, then a velocity discontinuity can be stabilized by introducing a uniform magnetic field [3–5]. The study of the Kelvin-Helmholtz instability is of much importance for the understanding of a variety of astrophysical phenomenon with plasma flow, such as the physics of the solar atmosphere, the stripping of gas from the clusters of moving galaxies, the structure of the tail of comets, and the interaction between the solar wind and the planetary magnetospheres and ionospheres [6, 7].

In the investigation of the hydromagnetic Kelvin-Helmholtz instability problem, many authors used hydromagnetic equations which assume that the Larmor radii of the charged particles (electrons and protons) are effectively zero. In several situations of astrophysical interest such as in interstellar and interplanetary plasmas, the approximation of zero Larmor radius in not valid, and it is also not realistic in a number of physical situations. It is known that the finite ion Larmor radius correction is important in many plasma and astrophysical situations [8–19]. The theory of finite Larmor radius stabilization of ideal magnetohydrodynamic modes has been developed over many years. The stability influence of finite Larmor radius corrections on plasma instabilities has been demonstrated by Rosenbluth *et al.* [20], Roberts and Taylor [21], Furth [22], Jukes [23] and many others. The problem of Kelvin-Helmholtz instability of a compressible as well as an incompressible plasma has been studied by Nagano [24] to include finite Larmor radius effect. He has also investigated the influence of finite Larmor radius on the Kelvin-Helmholtz instability of the magnetopause, and discussed the results in comparison with experimental

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observations. The finite Larmor radius effect on the instability of incompressible infinitely conducting superposed fluids has been considered by Karla [25]. He has shown that the finite Larmor radius stabilizes perturbations for all wavenumbers. The hydromagnetic Kelvin-Helmholtz and Rayleigh-Taylor instabilities with finite Larmor radius is also of interest in a variety of space, astrophysical and geophysical situations, and have been recently investigated for physical models involving plasma flows [26–28], thermal effects [29–37], and self-gravitating media [38–42], and references cited therein.

From the discussion of various stability problems which are relevant to certain astrophysical situations, the effect of suspended particles is widely considered as it has been observed that interstellar medium contain grains which are small particles formed in the outer atmosphere of stars and ejected into the medium. The effect of suspended particles on hydrodynamic stability problems is widely considered by many authors. Saffman [43] has studied in detail a dusty gas in magnetohydrodynamics. Scanlon and Segel [44] have made a thorough study of the implifications of suspended particles in hydromagnetics in the context of the Bénard convection problem. Sanghvi and Chhajlani [45] have incorporated the finite resistivity effect on the Rayleigh-Taylor configuration of a stratified plasma in the presence of suspended particles, and found that the particles have stabilizing as well as destabilizing influence under certain conditions. Palaniswamy and Purushotham [46] have studied the effect of shear flow of stratified fluids with fine dust and found that the effects of fine dust to increase the region of instability. For recent works about fluid-particle flows, see references [47–50]. On the other hand, the multiphase fluid systems are concerned with the motion of a liquid or gas containing immiscible inert identical particles. Of all multiphase fluid systems observed in nature, blood flows in arteries, flows in rocket tubes, dust in gas cooling systems to enhance heat transfer processes, movement of inert solid particles in atmosphere, sand or other particles in sea or ocean beaches are the most common examples of multiphase fluid systems [51]. For an excellent review about the subject, see the paper of Maury and Glowinski [52].

From the above discussion, it is obvious that the inclusion of suspended particles in the instability problem, together with the finite Larmor radius corrections, is of interest because of its relevance to certain astrophysical contexts [53–55]. Therefore, in the present study, we have incorporated finite ion Larmor radius and viscosity corrections in the Kelvin-Helmholtz configuration of two superposed streaming fluids acted upon by a uniform magnetic field transverse to the direction of streaming in the presence of suspended particles. We have discussed some interesting implifications of the finite Larmor radius corrections, viscosity, and suspended particles.

2 Formulation of the problem

We consider a model of two semi-infinite homogeneous viscous fluids separated by a plane interface at $z = 0$. Each of these regions $z < 0$ and $z > 0$, denoted by the subscripts 1 and 2, are permeated by a homogeneous distribution of suspended particles of the same density. Thus the medium can be regarded as a uniform mixture of gas and suspended particles. The gas is considered infinitely conducting and incompressible, while the particles are assumed to be non-conducting. Let the mixture of the hydromagnetic fluid and the suspended particles stream with velocity $\mathbf{U}(0, U, 0)$ in a transverse magnetic field $\mathbf{H}(H, 0, 0)$ which is essentially uniform, and it is acted upon by a downward gravitational field $\mathbf{g}(0, 0, -g)$.

In order to bring out the essential features of the problem, we shall make certain simplifying assumptions about the motion of the suspended particles. We assume that the particles are uniform in size and spherical in shape. Let $\mathbf{v}$ and N denote the velocity and the number density of the particles, respectively. It is supposed also that the bulk concentration of the particles is very small so that the net effect of the particles on the gas is equivalent to an extra body force $KN(\mathbf{v} - \mathbf{U})$, where K is a constant given by $K = 6\pi a\mu$ (Stokes drag formula), a being the particle radius, and μ is the viscosity of the clean gas.

Thus the relevant equations of motion and continuity of the hydromagnetic viscous fluid are

$$\rho\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U}\right] = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{U} - \nabla\cdot\mathbf{\Pi} + KN(\mathbf{v} - \mathbf{U}) + \frac{\mu_{\rm e}}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H} + \left(\frac{\partial w}{\partial \mathbf{x}} + \frac{\partial \mathbf{U}}{\partial z}\right)\frac{\mathrm{d}\mu}{\mathrm{d}z} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{U}) = 0 \tag{2}$$

where ρ and p denote the density and the pressure of the gas, respectively, $\mu_{\rm e}$ is the magnetic permeability which is assumed to be constant, μ is the dynamic fluid viscosity, and $\mathbf{x} = (x, y, z)$. The finite Larmor radius correction has been incorporated through the stress tensor $\mathbf{\Pi}$ in the equation of motion.

The incompressibility condition is

$$\nabla\cdot\mathbf{U} = 0. \tag{3}$$

Also, the Maxwell's equations for a perfect conductor are

$$\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H}\cdot\nabla)\mathbf{U} - (\mathbf{U}\cdot\nabla)\mathbf{H} \tag{4}$$

$$\nabla\cdot\mathbf{H} = 0. \tag{5}$$

The force exerts by the gas on the particles is equal and opposite to the force exerted by the particles on the gas. The buoyancy force on the particles is neglected, as its stabilizing effect is extremely small. Inter-particle distances are assumed to be very large as compared to the diameter of the particles, and so the inter-particle reactions can be ignored. Under these restrictions, the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = KN(\mathbf{U} - \mathbf{v}) \tag{6}$$

and

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\mathbf{v}) = 0 \tag{7}$$

where mN is the mass of the particles per unit volume.

3 Perturbation equations

To investigate the stability of this system, we assume the following perturbations in various physical quantities

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{u}', \ \ \mathbf{v} = \mathbf{v}_0 + \mathbf{v}',$$
$$N = N_0 + N', \mathbf{H} = \mathbf{H}_0 + \mathbf{h},$$
$$p = p_0 + \delta p, \ \text{and} \ \ \rho = \rho_0 + \delta\rho \tag{8}$$

where the quantities with the subscript 0 denote equilibrium values, and the quantities $\mathbf{u}'(u, v, w)$, $\mathbf{v}'$, N', $\mathbf{h}(h_x, h_y, h_z)$, δp, and $\delta\rho$ denote the perturbations in the gas velocity, the velocity of the particles, number density of the particles, magnetic field, fluid pressure, and the density of the gas, respectively.

Substituting equation (8) into equations (1–7), and neglecting the second and higher order terms with respect to small fluctuations, we obtain the following set of linearized equations for the considered system, after dropping the subscript 0 from the equilibrium quantities, and the primes from the perturbations quantities

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}\right] = -\nabla\delta p + \mathbf{g}\delta\rho + \mu\nabla^2\mathbf{u} - \nabla\cdot\mathbf{\Pi}$$
$$+ KN(\mathbf{v}-\mathbf{u}) + \frac{\mu_\mathrm{e}}{4\pi}(\nabla\times\mathbf{h})\times\mathbf{H} + \left(\frac{\partial w}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial z}\right)\frac{\mathrm{d}\mu}{\mathrm{d}z} \tag{9}$$

$$\left[1 + \tau\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\right]\mathbf{v} = \mathbf{u} \tag{10}$$

$$\frac{\partial \delta\rho}{\partial t} + (\mathbf{U}\cdot\nabla)\delta\rho + (\mathbf{u}\cdot\nabla)\rho = 0 \tag{11}$$

$$\frac{\partial \mathbf{h}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{h} = (\mathbf{H}\cdot\nabla)\mathbf{u} \tag{12}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{13}$$

and

$$\nabla\cdot\mathbf{h} = 0 \tag{14}$$

where $\tau = m/K$ denotes the relaxation time for the suspended particles.

The stress tensor $\mathbf{\Pi}$ has the following components for the horizontal magnetic field $\mathbf{H}(H, 0, 0)$ [56]

$$\Pi_{xx} = 0$$
$$\Pi_{yy} = -\Pi_{zz} = -\rho\nu_0\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$
$$\Pi_{xy} = \Pi_{yx} = -2\rho\nu_0\left(\frac{\partial u}{\partial z}\right)$$
$$\Pi_{xz} = \Pi_{zx} = 2\rho\nu_0\left(\frac{\partial u}{\partial y}\right)$$
$$\Pi_{zy} = \Pi_{yz} = \rho\nu_0\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)\cdot \tag{15}$$

The parameter ν_0 has the dimension but not the exact physical significance of a kinematic viscosity, and it is defined by $\nu_0 = R_\mathrm{L}^2\Omega_\mathrm{L}/4$, where R_L is the ion Larmor radius, and Ω_L is the ion gyrofrequency.

Let us look for solutions of the form

$$\exp(\mathrm{i}ky + nt) \tag{16}$$

where k is the wavenumber along the y-axis, and n is the growth rate of the perturbation

On eliminating $\mathbf{v}$ from equation (9) with the help of equation (10), and then employing equations (15, 16) on equations (9–14), we obtain the following set of equations

$$\rho(n+\mathrm{i}kU)\left[1 + \frac{\alpha_0}{\{1+\tau(n+\mathrm{i}kU)\}}\right]v = -\mathrm{i}k\delta p$$
$$+ 2\nu_0(D\rho)(Dw) + \rho\nu_0(D^2-k^2)w$$
$$+ \mu(D^2-k^2)v + (\mathrm{i}kw + Dv)D\mu \tag{17}$$

$$\rho(n+\mathrm{i}kU)\left[1 + \frac{\alpha_0}{\{1+\tau(n+\mathrm{i}kU)\}}\right]w = -D\delta p - g\delta\rho$$
$$-\nu_0(D\rho)(Dv) - \rho\nu_0(D^2-k^2)v - \mathrm{i}k\nu_0(D\rho)w$$
$$+ \mu(D^2-k^2)w + 2(D\mu)Dw \tag{18}$$

$$(n+\mathrm{i}kU)\delta\rho = -wD\rho \tag{19}$$

$$\mathrm{i}kv + Dw = 0 \tag{20}$$

$$\mathrm{i}kh_y + Dh_z = 0 \tag{21}$$

where $\alpha_0 = mN/\rho$ denotes the mass concentration of the particles, and $D = \mathrm{d}/\mathrm{d}z$.

Equations (17, 18) can be solved using equations (19−21) to obtain the following differential equation governing the perturbed velocity component w

$$(n+\mathrm{i}kU)\left[1 + \frac{\alpha_0}{\{1+\tau(n+\mathrm{i}kU)\}}\right]\left[D(\rho Dw) - k^2\rho w\right]$$
$$+ 2\mathrm{i}\nu_0 k\left[D(D\rho\, Dw) - k^2(D\rho)w\right] + \frac{gk^2w(D\rho)}{(n+\mathrm{i}kU)}$$
$$-\left[D\left\{\mu(D^2-k^2)Dw\right\} - k^2\mu(D^2-k^2)w\right]$$
$$-\left[D\left\{(D\mu(D^2+k^2)w\right\} - 2k^2(D\mu)(Dw)\right] = 0. \tag{22}$$

It should be noted here that the density of the suspended particles in the two regions $z < 0$ and $z > 0$ (denoted by the subscripts 1 and 2) is assumed to be the same.

4 The characteristic equation

We consider the case of two superposed viscous fluids of densities ρ_1 ($z < 0$) and ρ_2 ($z > 0$), kinematic viscosities ν_1 and ν_2 separated bt a horizontal boundary at $z = 0$. Let the velocities of streaming of the two fluids be $\mathbf{U}_1(0, U_1, 0)$ and $\mathbf{U}_2(0, U_2, 0)$. Thus in the regions of constant μ and ρ, the differential equation (22) becomes

$$(D^2 - k^2)(D^2 - q_j^2)w_j = 0, \quad j = 1, 2 \tag{23}$$

where

$$q_j^2 = k^2 + \frac{(n + \mathrm{i}kU_j)}{\nu_j}\left[1 + \frac{\alpha_j}{\{1 + \tau(n + \mathrm{i}kU)\}}\right]\cdot \tag{24}$$

Since w must be vanish both when $z \to -\infty$ (in the lower fluid) and $z \to \infty$ (in the upper fluid), then the general solution of equation (23) in the two regions can be written as

$$w_j = [A_j \exp(\pm kz) + B_j \exp(\pm q_j z)]\,(n + \mathrm{i}kU_j) \qquad j = 1, 2 \tag{25}$$

where A_1, B_1, A_2, and B_2 are constants of integration. Note that, in writing the solutions (25), it is assumed such that q_1 and q_2 are so defined that their real parts are positive.

The solutions (25) must satisfy certain boundary conditions. The boundary conditions to be satisfied at the interface $z = 0$ are

$$\frac{w_1}{\sigma_1} = \frac{w_2}{\sigma_2} \quad \text{at} \quad z = 0 \tag{26}$$

$$D\left(\frac{w_1}{\sigma_1}\right) = D\left(\frac{w_2}{\sigma_2}\right) \quad \text{at} \quad z = 0 \tag{27}$$

$$\mu_1(D^2 + k^2)\left(\frac{w_1}{\sigma_1}\right) = \mu_2(D^2 + k^2)\left(\frac{w_2}{\sigma_2}\right) \quad \text{at} \quad z = 0 \tag{28}$$

where $\sigma_j = (n + \mathrm{i}kU_j)$, $j = 1, 2$. Also integrating equation (22) across the interface $z = 0$, we obtain another boundary condition

$$\begin{aligned}&\Delta_0\left\{\rho(n + \mathrm{i}kU)Dw\left[1 + \frac{\alpha_0}{\{1 + \tau(n + \mathrm{i}kU)\}}\right]\right\} \\ &- \Delta_0[\mu(D^2 - k^2)Dw] - 2\mathrm{i}\nu_0 k^3 \Delta_0[\rho(n + \mathrm{i}kU)]\left(\frac{w}{n + \mathrm{i}kU}\right)_0 \\ &+ gk^2\Delta_0(\rho)\left(\frac{w}{n + \mathrm{i}kU}\right)_0 + 2k^2\Delta_0[\mu(n + \mathrm{i}kU)] \\ &\times\left(\frac{Dw}{n + \mathrm{i}kU}\right)_0 = 0 \quad \text{at} \quad z = 0\end{aligned} \tag{29}$$

where Δ_0 denotes the jump of a quantity experiences at the interface $z = 0$; w_0 and Dw_0 are the common values of w_1, w_2, and Dw_1, Dw_2, respectively, at $z = 0$.

Applying the boundary conditions (26–29) to the solutions (25), we obtain

$$A_1 + B_1 = A_2 + B_2 \tag{30}$$

$$kA_1 + q_1 B_1 = -kA_2 - q_2 B_2 \tag{31}$$

$$\beta_1\nu_1\left[2k^2 A_1 + (q_1^2 + k^2)B_1\right] = \beta_2\nu_2 \times\left[2k^2 A_2 + (q_2^2 + k^2)B_2\right] \tag{32}$$

and

$$\begin{aligned}&\left\{\beta_1\sigma_1^2\left[1 + \frac{\alpha_1}{(1 + \tau\sigma_1)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1)\right. \\ &\left.+ k^2(\beta_1\nu_1\sigma_1 - \beta_2\nu_2\sigma_2) + \frac{gk}{2}(\beta_1 - \beta_2)\right\}A_1 + \left\{\frac{q_1\beta_1\sigma_1^2}{k}\right. \\ &\times\left[1 + \frac{\alpha_1}{(1 + \tau\sigma_1)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1) = q_1 k(\beta_1\nu_1\sigma_1 \\ &\left.- \beta_2\nu_2\sigma_2) - (q_1\beta_1\nu_1\sigma_1/k)(q_1^2 - k^2) + \frac{gk}{2}(\beta_1 - \beta_2)\right\}B_1 \\ &+ \left\{\beta_2\sigma_2^2\left[1 + \frac{\alpha_2}{(1 + \tau\sigma_2)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1)\right. \\ &\left.-k^2(\beta_1\nu_1\sigma_1 - \beta_2\nu_2\sigma_2) + \frac{gk}{2}(\beta_1 - \beta_2)\right\}A_2 \\ &+ \left\{\frac{q_2\beta_2\sigma_2^2}{k}\left[1 + \frac{\alpha_2}{(1 + \tau\sigma_2)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1)\right. \\ &= q_2 k(\beta_1\nu_1\sigma_1 - \beta_2\nu_2\sigma_2) - (q_2\beta_2\nu_2\sigma_2/k)\left(q_2^2 - k^2\right) \\ &\left.+\frac{gk}{2}(\beta_1 - \beta_2)\right\}B_2 = 0\end{aligned} \tag{33}$$

where we have written

$$\alpha_j = \frac{mN}{\rho_j} \quad \text{and} \quad \beta_j = \frac{\rho_j}{(\rho_1 + \rho_2)}, \quad j = 1, 2. \tag{34}$$

Eliminating A_1, B_1, A_2, and B_2 from equations (30–33), we obtain

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ k & q_1 & k & q_2 \\ 2k^2\beta_1\nu_1 & \beta_1\nu_1(q_1^2 + k^2) & -2k^2\beta_2\nu_2 & -\beta_2\nu_2(q_2^2 + k^2) \\ a_1 & b_1 & a_2 & b_2 \end{vmatrix} = 0 \tag{35}$$

where

$$\begin{aligned}a_j = \beta_j\sigma_j^2\left[1 + \frac{\alpha_j}{(1 + \tau\sigma_j)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1) \\ \pm k^2(\beta_1\nu_1\sigma_1 - \beta_2\nu_2\sigma_2) + \frac{gk}{2}(\beta_1 - \beta_2)\end{aligned} \tag{36}$$

$$\begin{aligned}b_j = \frac{q_j\beta_j\sigma_j^2}{k}\left[1 + \frac{\alpha_j}{(1 + \tau\sigma_j)}\right] + \mathrm{i}\nu_0 k^2(\beta_2\sigma_2 - \beta_1\sigma_1) \\ \pm q_j k(\beta_1\nu_1\sigma_1 - \beta_2\nu_2\sigma_2) - \frac{q_j\beta_j\nu_j\sigma_j}{k}(q_j^2 - k^2) + \frac{gk}{2}(\beta_1 - \beta_2).\end{aligned} \tag{37}$$

The determinant (35) can be reduced by subtracting the first column from the second, the third column from the fourth, and adding the first column to the third. By this procedure, we obtain

$$\begin{vmatrix} q_1-k & 2k & q_2-k \\ \beta_1\nu_1(q_1^2-k^2) & 2k^2(\beta_1\nu_1-\beta_2\nu_2) & -\beta_2\nu_2(q_2^2-k^2) \\ b_1-a_1 & a_1+a_2 & b_2-a_2 \end{vmatrix} = 0. \quad (38)$$

Evaluating the determinant (38), we obtain the following characteristic equation for the composite gas particle streaming viscous medium

$$\begin{aligned} &-2k\left\{k(\beta_1\nu_1-\beta_2\nu_2)(q_1-k)-\beta_1\sigma_1\left[1+\frac{\alpha_1}{(1+\tau\sigma_1)}\right]\right\} \\ &\times\left\{\beta_2\sigma_2^2\left[1+\frac{\alpha_2}{(1+\tau\sigma_2)}\right]+k(q_2-k)(\beta_1\nu_1\sigma_1-\beta_2\nu_2\sigma_2)\right\} \\ &+2k\left\{k(\beta_1\nu_1-\beta_2\nu_2)(q_2-k)+\beta_2\sigma_2\left[1+\frac{\alpha_2}{(1+\tau\sigma_2)}\right]\right\} \\ &\times\left\{\beta_1\sigma_1^2\left[1+\frac{\alpha_1}{(1+\tau\sigma_1)}\right]-k(q_1-k)(\beta_1\nu_1\sigma_1-\beta_2\nu_2\sigma_2)\right\} \\ &+\left\{\beta_1\sigma_1(q_2-k)\left[1+\frac{\alpha_1}{(1+\tau\sigma_1)}\right]+\beta_2\sigma_2(q_1-k)\right. \\ &\left.\times\left[1+\frac{\alpha_2}{(1+\tau\sigma_2)}\right]\right\}\left\{\beta_1\sigma_1^2\left[1+\frac{\alpha_1}{(1+\tau\sigma_1)}\right]\right. \\ &+2\mathrm{i}\nu_0k^2(\beta_2\sigma_2-\beta_1\sigma_1)+\beta_2\sigma_2^2 \\ &\left.\times\left[1+\frac{\alpha_2}{(1+\tau\sigma_2)}\right]+gk(\beta_1-\beta_2)\right\}=0. \quad (39) \end{aligned}$$

The characteristic equation (39) represents the combined influence of the kinematic viscosity, the suspended particles, and the finiteness of ion Larmor radius on the hydromagnetic Kelvin-Helmholtz instability of two superposed viscous fluids.

5 Dispersion relation and discussion

The characteristic relation (39) is quite complicated as the values of $q_1$and q_2 involve square roots. We therefore make the assumption that the two fluids are highly viscous. Under this assumption, equation (24) yields

$$q_j = k+\frac{\sigma_j}{2\nu_j k}\left[1+\frac{\alpha_j}{(1+\tau\sigma_j)}\right], \; j=1,2 \quad (40)$$

substituting the values of q_1-k and q_1-k from equation (40) in equation (39), and putting $\nu_1=\nu_2=\nu$ (the case of equal kinematic viscosities, for mathematical simplicity, as in Ref. [1]), we obtain the following dispersion relation

$$\begin{aligned} &\tau^2n^4+[2\nu k^2\tau^2+\tau(2+\alpha_1\beta_1+\alpha_2\beta_2)+\mathrm{i}k\tau^2\{U_1(1+2\beta_1) \\ &+U_2(1+2\beta_2)\}+2\mathrm{i}\nu_0k^2\tau^2(\beta_2-\beta_1)]n^3+[1+\alpha_1\beta_1+\alpha_2\beta_2 \\ &+4\nu k^2\tau+2\mathrm{i}\nu k^3\tau^2\{U_1(1+\beta_1)+U_2(1+\beta_2)\}+\mathrm{i}k\tau\{U_1(1+4\beta_1 \\ &+2\alpha_1\beta_1+\alpha_2\beta_2)+U_2(1+4\beta_2+\alpha_1\beta_1+2\alpha_2\beta_2)\}-3k^2\tau^2(\beta_1U_1^2 \\ &+\beta_2U_2^2+U_1U_2)+4\mathrm{i}\nu_0k^2\tau(\beta_2-\beta_1)+2\nu_0k^3\tau^2\{U_1(2\beta_1-\beta_2) \\ &+U_2(\beta_1-2\beta_2)\}+\tau^2gk(\beta_1-\beta_2)]n^2+[2\nu k^2+2\mathrm{i}\nu k^3\tau\{U_1(1 \\ &+2\beta_1)+U_2(1+2\beta_2)\}-2\nu k^4\tau^2(\beta_1U_1^2+\beta_2U_2^2+2U_1U_2) \\ &+2\mathrm{i}k\{\beta_1U_1(1+\alpha_1)+\beta_2U_2(1+\alpha_2)\}-k^2\tau\{\beta_1U_1^2(4+\alpha_1) \\ &+\beta_2U_2^2(4+\alpha_2)+2(1+\alpha_1\beta_1+\alpha_2\beta_2)U_1U_2\}-\mathrm{i}k^3\tau^2\{\beta_1U_1^3 \\ &+\beta_2U_2^3+3\beta_1U_1^2U_2+3\beta_2U_1U_2^2\}+2\mathrm{i}\nu_0k^2(\beta_2-\beta_1) \\ &-2\nu_0k^3\tau\{(\beta_2-3\beta_1)U_1+(3\beta_2-\beta_1)U_2\}-2\nu_0k^4\tau^2\{2(\beta_2 \\ &-\beta_1)U_1U_2+\beta_2U_2^2-\beta_1U_1^2\}+2\tau gk(\beta_1-\beta_2)+\mathrm{i}\tau^2gk^2(U_1 \\ &+U_2)(\beta_1-\beta_2)]n+[\{2\mathrm{i}\nu k^3(\beta_1U_1+\beta_2U_2)-k^2(\beta_1U_1^2+\beta_2U_2^2) \\ &-2\nu_0k^3(\beta_2U_2-\beta_1U_1)+gk(\beta_1-\beta_2)\}\{1+\mathrm{i}k\tau(U_1+U_2) \\ &-k^2\tau^2U_1U_2\}-k^2(\alpha_1\beta_1U_1^2+\alpha_2\beta_2U_2^2)-\mathrm{i}k^3\tau U_1U_2(\alpha_1\beta_1U_1 \\ &+\alpha_2\beta_2U_2)]=0. \quad (41) \end{aligned}$$

In the absence of viscosity ($\nu=0$). Equation (41) reduces to the dispersion relation obtained earlier by Sanghvi and Chhajlani [56] (except Sect. 5.2, in our limiting case of vanishing of both finite Larmor radius and fluid velocities), and their results are therefore corrected and recovered here. In the absence of viscosity and suspended particles ($\nu=0$, $\tau=0$, $\alpha_1=\alpha_2=0$), this equation reduces to the dispersion relation obtained by Singh and Hans [57]. Note that, the effect of suspended particles enters into the dispersion relation (41) through two parameters α_j and τ measuring the mass concentration and the relaxation time of the particles. We shall now discuss the following cases of interest.

5.1 General configuration

In this subsection, we treat the configuration in the absence of suspended particles and finite Larmor radius effect. The dispersion relation (41) for this case, on substituting ($\nu_0=0$, $\tau=0$, $\alpha_1=\alpha_2=0$) reduces to

$$\begin{aligned} &n^2+[2\nu k^2+2\mathrm{i}k(\beta_1U_1+\beta_2U_2)]n+[gk(\beta_1-\beta_2) \\ &-k^2(\beta_1U_1^2+\beta_2U_2^2)+2\mathrm{i}\nu k^2(\beta_1U_1+\beta_2U_2)]=0. \quad (42) \end{aligned}$$

The roots of equation (42) are given by

$$\begin{aligned} n=&-[\nu k^2+\mathrm{i}k(\beta_1U_1+\beta_2U_2)] \\ &\pm\left[k^2\beta_1\beta_2(U_1-U_2)^2+\nu^2k^4-gk(\beta_1-\beta_2)\right]^{1/2}. \quad (43) \end{aligned}$$

When $\beta_1 > \beta_2$ (stable Kelvin-Helmholtz case), we find from equation (43) that the Kelvin-Helmholtz instability is suppressed if

$$k^2\beta_1\beta_2(U_1 - U_2)^2 < gk(\beta_1 - \beta_2) \tag{44}$$

or if

$$\left[k^2\beta_1\beta_2(U_1 - U_2)^2 + \nu^2k^4\right] \leq gk(\beta_1 - \beta_2) \tag{45}$$

since under the above restriction, equation (42) will not allow any real positive root of n, which implies the stability of the system. Note that the kinematic viscosity has a stabilizing effect in this case, Thus we conclude that the considered Kelvin-Helmholtz configuration is stabilized for the wavenumbers determined by the inequalities (44) or (45). Also we find that instability results for all wavenumbers satisfying the condition

$$k^2\beta_1\beta_2(U_1 - U_2)^2 > gk(\beta_1 - \beta_2). \tag{46}$$

From equations (44, 46), it follows that the system is stable or unstable according as

$$k^2\beta_1\beta_2(U_1 - U_2)^2 \lesseqgtr gk(\beta_1 - \beta_2) \tag{47}$$

therefore, there exists a critical wavenumber k_c given by

$$k_c = \frac{g(\beta_1 - \beta_2)}{\beta_1\beta_2(U_1 - U_2)^2} \tag{48}$$

such that the system is stable when $k < k_c$, and it is otherwise unstable.

When $\beta_1 < \beta_2$ (unstable Kelvin-Helmholtz case), it is easy to see from equation (43) that the Kelvin-Helmholtz configuration remains always unstable as one of the roots of equation (42) is complex with positive real part. It is clear from the above analysis that interchange perturbations remain unaffected by the presence of a magnetic field.

It is elucidating to consider the case of two streaming viscous fluids in the absence of gravitational force ($g = 0$), for which we have

$$n = -[\nu k^2 + \mathrm{i}k(\beta_1 U_1 + \beta_2 U_2)] \pm \left[k^2\beta\beta_2(U_1 - U_2)^2 + \nu^2k^4\right]^{1/2} \tag{49}$$

which means that the system is always unstable irrespective of the magnitude and direction of the streaming velocities, or the viscosity of the fluid.

Consider now the case of non-streaming fluids ($U_1 = U_2 = 0$) under gravity, one obtains from equation (43)

$$n = -\nu k^2 \pm \left[\nu^2k^4 - gk\left(\beta_1 - \beta_2\right)\right]^{1/2}. \tag{50}$$

When $\beta_1 > \beta_2$, we find from equation (50) that the Rayleigh-Taylor instability is suppressed for all the wavenumbers values. When $\beta_1 < \beta_2$, in this case, it is easy also to see from equation (50) that the Rayleigh-Taylor configuration remains always unstable, and that the viscosity has a destabilizing effect in this case. Finally, note that in the absence of kinematic viscosity, then equation (50) shows that the system is stable or unstable according to $\beta_1 > \beta_2$ or $\beta_1 < \beta_2$, respectively.

5.2 Static configuration

In this subsection, we shall deal with the case of non-streaming superposed hydromagnetic viscous fluids of different densities in the presence of suspended particles. In order to discuss implification of the presence of particles, we analyze the case for vanishing finite Larmor radius ($\nu_0 = 0$, $U_1 = U_2 = 0$). In this case the dispersion equation (41) can be written as

$$\tau^2n^4 + [2\nu k^2\tau^2 + \tau(2 + \alpha_1\beta_1 + \alpha_2\beta_2)]n^3 + [1 + \alpha_1\beta_1 + \alpha_2\beta_2 + 4\nu k^2\tau + \tau^2 gk(\beta_1 - \beta_2)]n^2 + [2\nu k^2 + 2\tau gk(\beta_1 - \beta_2)]n + gk(\beta_1 - \beta_2) = 0. \tag{51}$$

Note that the dispersion relation (51) is different from the dispersion relation (35) obtained earlier by Sanghvi and Chhajlani [56] due to an error in algebra (see their Eqs. (27, 35)). This error is corrected here, and we will discuss the stability conditions in view of the corrected dispersion relation (51). Introducing the relaxation frequency parameter $f(= 1/\tau)$ of the suspended particles and simplifying the above equation, we obtain

$$n^4 + n^3[2\nu k^2 + f(2 + \alpha')] + n^2[f^2(1 + \alpha') + 4\nu k^2 f + gk(\beta_1 - \beta_2)] + 2n[\nu k^2 f^2 + fgk(\beta_1 - \beta_2)] + f^2 gk(\beta_1 - \beta_2) = 0 \tag{52}$$

where $\alpha' = mN/(\rho_1 + \rho_2)$. We can distinguish now between the following two cases

(i) Clean configuration stable ($\beta_1 > \beta_2$), we find, using a necessary condition of the Hurwitz criterion, that equation (52), which has all the coefficients are positive and real, does not admit any real positive or complex root with positive real part, implying the stability of the considered system. Thus the stable configuration remains stable even in the presence of suspended particles and fluid viscosity. We must note here that this condition is necessary but it is not a sufficient one for which positivity of Hurwitz's determinants has to be proved [58]. We may conclude that the considered configuration is stable when $\beta_1 > \beta_2$.

(ii) Clean configuration unstable ($\beta_1 < \beta_2$), when the upper fluid is heavier than the lower one, then equation (52) will necessarily possess one real positive root n_0, which leads to an instability of the system.

To examine the behaviour of the growth rate with increasing relaxation frequency of the suspended particles, we need to calculate $\mathrm{d}n_0/\mathrm{d}f$ from equation (52), to give

$$\frac{\mathrm{d}n_0}{\mathrm{d}f} = -[n_0^3(2 + \alpha') + 2n_0^2\{f(1 + \alpha') + 2\nu k^2\} + 2n_0\{2\nu k^2 f + gk(\beta_1 - \beta_2)\} + 2fgk(\beta_1 - \beta_2)][4n_0^3 + 3n_0^2\{2\nu k^2 + f(2 + \alpha')\} + 2n_0\{f^2(1 + \alpha') + 4\nu k^2 f + gk(\beta_1 - \beta_2)\} + 2\{\nu k^2 f^2 + fgk(\beta_1 - \beta_2)\}]^{-1}. \tag{53}$$

The growth rate turns out to be negative if the upper or lower signs of the inequalities

$$n_0^3(2+\alpha') + 2n_0^2\{f(1+\alpha') + 2\nu k^2\} + 4n_0\nu k^2 f \gtrless 2gk(\beta_2 - \beta_1)(n_0 + f) \quad (54)$$

and

$$4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha')\} + 2n_0\{f^2(1+\alpha') + 4\nu k^2 f\} + 2\nu k^2 f^2 \gtrless 2gk(\beta_2 - \beta_1)(n_0 + f) \quad (55)$$

hold simultaneously. Therefore, we conclude that the growth rate of the unstable Rayleigh-Taylor modes is decreased with increasing relaxation frequency of the suspended particles. This means, under the restrictions (54, 55), that the particles have stabilizing influence on the configuration. Also, if the upper sign of inequality (54) and the lower sign of inequality (55), or *vice versa* hold simultaneously, then the suspended particles have destabilizing effect on the considered system in this case.

Similarly, the behaviour of the growth rate with increasing kinematic viscosity can be obtained from equation (52), where

$$\frac{\mathrm{d}n_0}{\mathrm{d}\nu} = -2n_0k^2(n_0+f)^2[4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha')\} + 2n_0\{f^2(1+\alpha') + 4\nu k^2 f + gk(\beta_1 - \beta_2)\} + 2\{\nu k^2 f^2 + fgk(\beta_1 - \beta_2)\}]^{-1} \quad (56)$$

therefore, we conclude that the growth rate of the unstable Rayleigh-Taylor modes is decreased or increased with increasing fluid viscosity if the denominator in equation (56) is positive or negative, respectively, *i.e.* if the upper or lower sign of inequality (55) holds, respectively. This means, under the restriction (55), that the fluid viscosity has a stabilizing as well as a destabilizing effect on the configuration.

The dispersion relation (52) can be written in dimensionless form by the substitutions $\widehat{n} = n/\sqrt{gk}$, $\widehat{f} = f/\sqrt{gk}$, $\widehat{\nu} = \nu/(gk)^{3/2}$, and $\widehat{k} = k\sqrt{gk}$. Thus we get

$$\widehat{n}^4 + \widehat{n}^3[2\widehat{\nu}\widehat{k}^2 + \widehat{f}(2+\alpha')] + \widehat{n}^2[\widehat{f}^2(1+\alpha') + 4\widehat{\nu}\widehat{k}^2\widehat{f} + (\beta_1 - \beta_2)] + 2\widehat{n}[\widehat{\nu}\widehat{k}^2\widehat{f}^2 + \widehat{f}(\beta_1 - \beta_2)] + \widehat{f}^2(\beta_1 - \beta_2) = 0 \quad (57)$$

where $\alpha' = KN\tau/(\rho_1 + \rho_2)$.

As an illustration of both the implifications of suspended particles and the fluid viscosity on the stability of the Rayleigh-Taylor mode, we have numerically solved equation (57) for positive real roots for various values of the non-dimensional parameters $\widehat{f}$ (or $\widehat{\nu}$) and α' (which characterize the influence of suspended particles). Figures 1–3 clearly show that the growth rate $\widehat{n}$ (as a function of the wavenumber $\widehat{k}$) decreases with increasing each of relaxation frequency $\widehat{f}$ of the suspended particles,

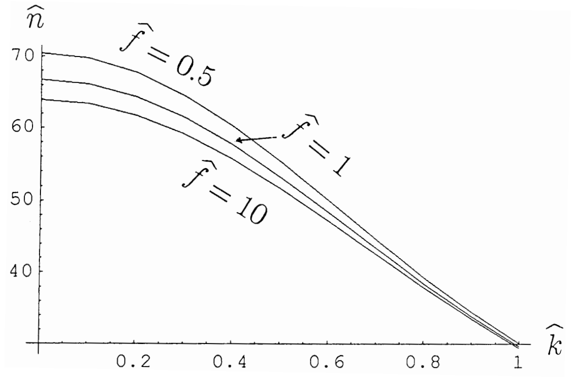

Fig. 1. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{f} = 0.5$, 1, and 10, with $\alpha' = 0.5$, $\widehat{\nu} = 0.8$, $\beta_1 = 0.2$, and $\beta_2 = 0.8$.

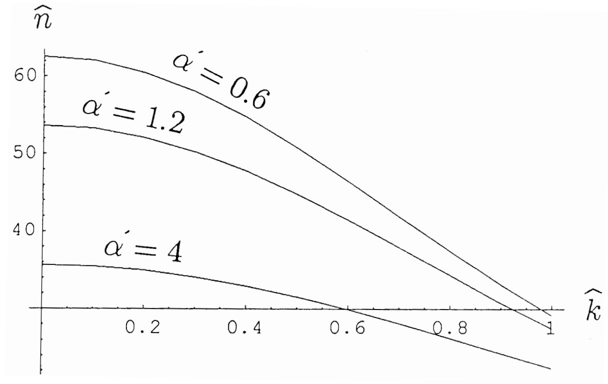

Fig. 2. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\alpha' = 0.6$, 1.2, and 4, with $\widehat{f} = 5$, $\widehat{\nu} = 0.8$, $\beta_1 = 0.2$, and $\beta_2 = 0.8$.

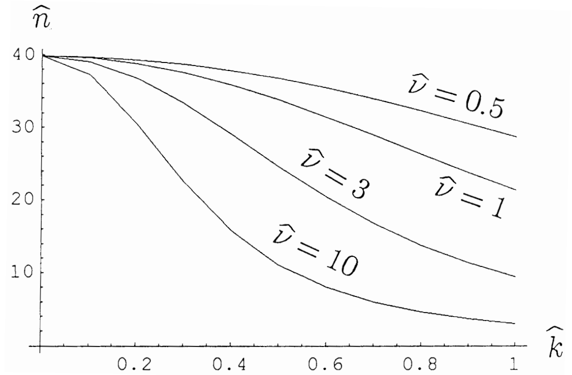

Fig. 3. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{\nu} = 0.5$, 1, 3 and 10, with $\widehat{f} = 5$, $\alpha' = 3$, $\beta_1 = 0.2$, and $\beta_2 = 0.8$.

the density α' of the particles as well as the fluid viscosity $\widehat{\nu}$, respectively, where we have kept $\beta_2 - \beta_1 = 0.6$ in these figures. It may be remarked here that an increase of $\widehat{f}(= 6\pi\mu a/m)$ implies an increase in the size of the particles (a), as we have assumed μ and m to be constants. Thus we conclude that the increasing of fluid viscosity, relaxation frequency of the particles (and their density) have stabilizing influences on the considered Rayleigh-Taylor configuration. In other words, as the size of the particles (of constant mass) increases, the growth rate of unstable Rayleigh-Taylor modes decreases even in the presence of fluid viscosity.

5.3 K-H instability including FLR and suspended particles

To apprehend implifications of finite Larmor radius corrections on the Kelvin-Helmholtz instability, we consider two viscous streaming fluids in the presence of a uniform magnetic field transverse to the direction of streaming, neglecting the effect of particles. We assume the streaming velocities of the two fluids to be U_1 and U_2, their densities being equal. The general dispersion relation (41) in this case becomes

$$n^2 + [2\nu k^2 + \mathrm{i}k(U_1 + U_2)]n + [\mathrm{i}\nu k^3(U_1 + U_2) + \nu_0 k^3(U_1 - U_2) - (k^2/2)(U_1^2 + U_2^2)] = 0. \quad (58)$$

The solution of the above dispersion relation is

$$n = -\frac{1}{2}[2\nu k^2 + \mathrm{i}k(U_1 + U_2)] \pm\sqrt{\nu^4 k^4 + \frac{k^2}{4}(U_1 - U_2)^2 - \nu_0 k^3(U_1 - U_2)}. \quad (59)$$

If $U_1 > U_2$ (the lower fluid is streaming faster than the upper fluid), the motion is stabilized or destabilized according as

$$k \gtrless \frac{(U_1 - U_2)}{4\nu_0} \quad (60)$$

respectively. Also, equation (59) shows that the medium is stabilizes for wavenumbers determined by the inequality

$$\left[\nu^4 k^2 + \frac{1}{4}(U_1 - U_2)^2\right] \le \nu_0 k(U_1 - U_2) \quad (61)$$

i.e.

$$k \le \frac{(U_1 - U_2)}{2\nu^2}\left[\nu_0 + \sqrt{\nu_0^2 - \nu^2}\right]. \quad (62)$$

Thus the medium is stable for wavenumbers $k \le k_\mathrm{c}$, where

$$k_\mathrm{c} = \frac{(U_1 - U_2)}{2\nu^2}\left[\nu_0 + \sqrt{\nu_0^2 - \nu^2}\right] \quad (63)$$

the kinematic viscosity is found to has a stabilizing effect in this case.

Hence, we conclude that the finite Larmor radius tends to stabilize the configuration and the critical wavenumber depends upon the relative velocity of the two fluids. On the contrary, if $U_1 < U_2$, the system is always unstable in this case. Furthermore, if we solve the dispersion equation (58) corresponding to fluids of equal densities and of streaming velocities U and $-U$ in the absence of suspended particles, we obtain the relation

$$n^2 + 2\nu k^2 n + k^2 U[2\nu_0 k - U] = 0. \quad (64)$$

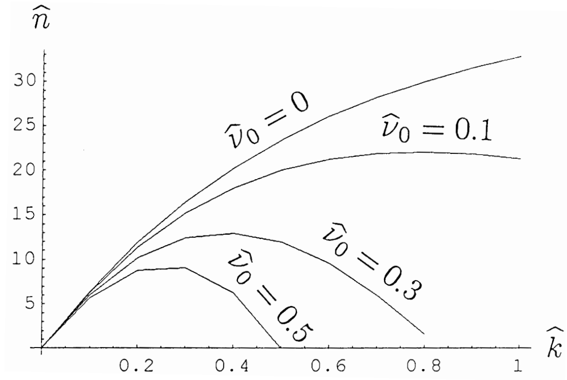

Fig. 4. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{\nu}_0 = 0$, 0.1, 0.3, and 0.5, with $\widehat{U} = 0.5$, and $\widehat{\nu} = 0.6$.

It corresponds to the Kelvin-Helmholtz model when there is a tangential discontinuity in velocity in a uniform plasma. This result, in the absence of viscosity, has been obtained by Karla [25], and also by Nagano [24]. In this case, we observe that the finite Larmor radius has a stabilizing influence as it reduces the frequency of oscillation of the system. The criterion for instability is

$$U > 2\nu_0 k \quad (65)$$

which yields the critical wavenumber

$$k_\mathrm{c} = \frac{U}{2\nu_0}\cdot \quad (66)$$

Here, we note that the finite Larmor radius effect stabilizes the perturbations for $k > k_\mathrm{c}$. In comparing this result with equation (62), we find that the critical wavenumbers are different (or not) for fluids streaming with different velocities (U_1, U_2), or with the speeds (U, 0) such that $\nu = \nu_0$, respectively.

Now we solve equation (64) numerically for positive roots in order to illustrate the influence of finite Larmor radius and fluid viscosity on the Kelvin-Helmholtz configuration. Note that we are analyzing interchange ($\mathbf{k} \perp \mathbf{H}$) perturbations. To do this, we put equation (64) in the non-dimensional form

$$\widehat{n}^2 + 2\widehat{\nu}\widehat{k}^2\widehat{n} + \widehat{k}^2\widehat{U}[2\widehat{\nu}_0\widehat{k} - \widehat{U}] = 0 \quad (67)$$

where we have introduced the non-dimensional parameters $\widehat{n} = nL/V$, $\widehat{k} = kL$, $\widehat{U} = U/V$, $\widehat{\nu}_0 = \nu_0/VL$, and $\widehat{\nu} = \nu/VL$, where V and L denote the Alfvén speed and the characteristic length, respectively. Figure 4 shows the variation of the growth rate $\widehat{n}$ (positive real part) as a function of the wavenumber $\widehat{k}$ for different values of $\widehat{\nu}_0$, taking $\widehat{U} = 0.5$ (fixed). The upper curve where $\widehat{\nu}_0 = 0$ correspond to the case of ideal magnetohydrodynamics in the presence of fluid viscosity. We notice that, for some value of $\widehat{\nu}_0$, the growth rate first increases for small $\widehat{k}$, attains a maximum but remains smaller than the usual magnetohydrodynamic approximation, and therefore decreases and becomes zero for the corresponding critical

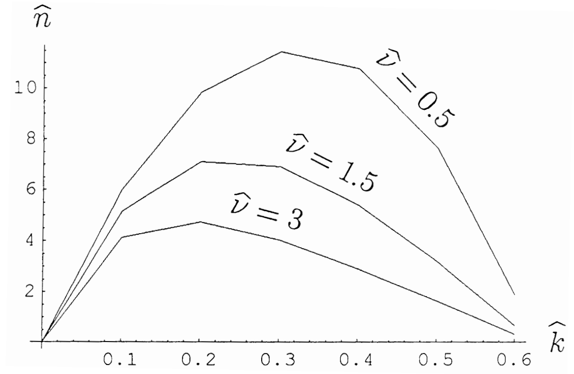

Fig. 5. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{\nu} = 0.5$, 1.5, and 3, with $\widehat{U} = 0.5$, and $\widehat{\nu}_0 = 0.4$.

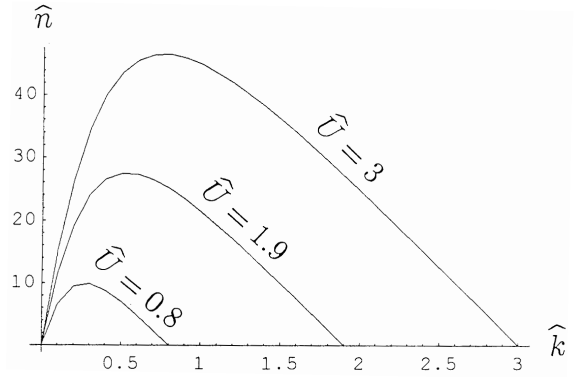

Fig. 6. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{U} = 0.8$, 1.9, and 3, with $\widehat{\nu} = 0.2$, and $\widehat{\nu}_0 = 0.5$.

wavenumber. The forgoing analysis indicates that interchange perturbations are stabilized by the finite Larmor radius effect for $\widehat{k} > \widehat{k}_c$, where $\widehat{k}_c = \widehat{U}/2\widehat{\nu}_0$. Furthermore, it is observed that as the finite Larmor radius increases, the domain of instability is also reduces. Also, equation (67) shows, in the absence of both $\widehat{\nu}$ and $\widehat{\nu}_0$, that the growth rate $\widehat{n}$ varies linearly with the wavenumber $\widehat{k}$ (*i.e.* $\widehat{n}$ increase with increasing $\widehat{k}$), but when $\widehat{\nu} \neq 0$, then he growth rate $\widehat{n}$ varies nonlinearly with the wavenumber $\widehat{k}$ (*i.e.* $\widehat{n}$ increase with increasing $\widehat{k}$ and attains a maximum, after which it decreases and becomes zero at $\widehat{k} = \widehat{k}_c$). This result shows the stabilizing effect of the fluid viscosity on the considered system, and it is also confirmed by Figure 5, in the wavenumbers range $0 \leq \widehat{k} \leq 0.6$ (approximately). Figure 6 shows the variation of the growth rate $\widehat{n}$ (positive real part) as a function of the wavenumber $\widehat{k}$ for different values of $\widehat{U}$. We notice that the growth rate first for small $\widehat{k}$, attains a maximum and therefore decreases and becomes zero for the corresponding critical wavenumber $\widehat{k}$. It is also observed that, as the fluid velocity increases, the domain of instability is also enlarged. Thus, the fluid velocity has a destabilizing influence on the considered system.

Finally, to study the combined influence of the finite Larmor radius, kinematic viscosity, and suspended particles, we specialize the dispersion relation (41) when identical gas particle composite medium occupy the two regions $z < 0$ and $z > 0$. The streaming velocities in the two viscous regions are assumed to be U and $-U$, in the presence of a uniform magnetic field transverse to the direction of streaming. In this case we have to put the following values in the dispersion relation (41): $\alpha_1 = \alpha_2 = \alpha_0$, $\beta_1 = \beta_2 = 1/2$, $U_1 = U$, and $U_2 = -U$. Thus we obtain the following dispersion relation

$$\tau^2 n^4 + \tau[2\nu k^2\tau + (2+\alpha_0)]n^3 + [1+\alpha_0+4\nu k^2\tau+2\nu_0 k^3\tau^2 U]n^2 + \{2\nu k^2(1+k^2\tau^2U^2) + k^2\tau U^2(\alpha_0-2) + 4\nu_0 k^3\tau U\}n + k^2U[\{2\nu_0 k - U\}\{1+k^2\tau^2U^2\} - \alpha_0 U] = 0. \quad (68)$$

The above dispersion relation, on writing $f = 1/\tau$ (the relaxation frequency parameter of the suspended particles) becomes

$$n^4 + [2\nu k^2 + f(2+\alpha_0)]n^3 + [f^2(1+\alpha_0) + 4\nu k^2 f + 2\nu_0 k^3 U]n^2 + \{2\nu k^2(f^2+k^2U^2) + k^2 f U^2(\alpha_0-2) + 4\nu_0 k^3 f U\}n + k^2U[\{2\nu_0 k - U\}\{f^2+k^2U^2\} - \alpha_0 f^2 U] = 0. \quad (69)$$

The dispersion relation (69) enables us to examine the transverse Kelvin-Helmholtz configuration in the presence of finite Larmor radius effect, and suspended particles simultaneously. We calculate the derivative of the growth rate of the unstable Kelvin-Helmholtz mode n_0 with increasing the finite Larmor radius ν_0, and the kinematic viscosity ν, respectively. From equation (69), we obtain

$$\frac{\mathrm{d}n_0}{\mathrm{d}\nu_0} = -\frac{2k^3U[n_0^2 + n_0 f + f^2 + k^2U^2]}{4n_0^3 + a_1 n_0^2 + a_2 n_0 + a_3} \quad (70)$$

$$\frac{\mathrm{d}n_0}{\mathrm{d}\nu} = -\frac{2n_0 k^2[(n_0+f)^2 + k^2U]}{4n_0^3 + a_1 n_0^2 + a_2 n_0 + a_3} \quad (71)$$

where

$$a_1 = 3[2\nu k^2 + f(2+\alpha_0)] \quad (72)$$

$$a_2 = 2[f^2(1+\alpha_0) + 4\nu k^2 f + 2\nu_0 k^3 U] \quad (73)$$

$$a_3 = k^2[2\nu(f^2+k^2U) + fU^2(\alpha_0-2) + 4\nu_0 k f U]. \quad (74)$$

The growth rate is negative if ($\alpha_0 \geq 2$), or if

$$[2\nu(f^2+k^2U) + \alpha_0 f U^2 + 4\nu_0 k f U] > 2fU^2 \quad (75)$$

and it will be positive if

$$[4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha_0)\} + 2n_0\{f^2(1+\alpha_0) + 4\nu k^2 f + 2\nu_0 k^3 U\} + k^2\{2\nu(f^2+k^2U) + fU^2(\alpha_0-2) + 4\nu_0 k f U\}] < 0. \quad (76)$$

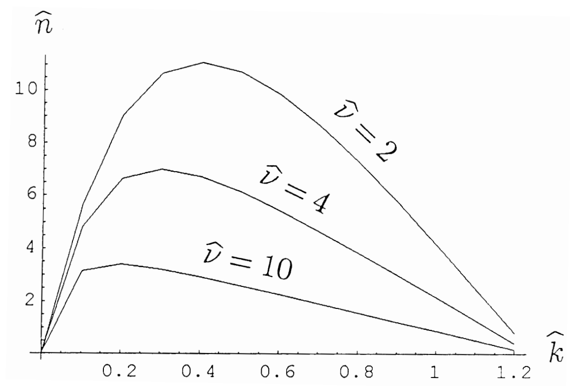

Fig. 7. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{\nu} = 2$, 4, and 10, with $\alpha_0 = 0.8$, $\widehat{U} = 0.7$, $\widehat{f} = 6$, and $\widehat{\nu}_0 = 0.5$.

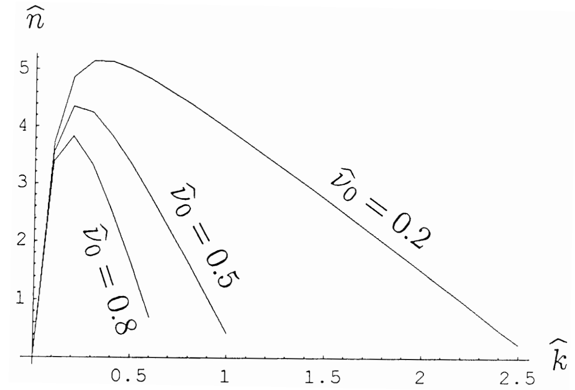

Fig. 8. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{\nu}_0 = 0.2$, 0.5, and 0.8, with $\alpha_0 = 0.8$, $\widehat{U} = 0.6$, $\widehat{f} = 5$, and $\widehat{\nu} = 5$.

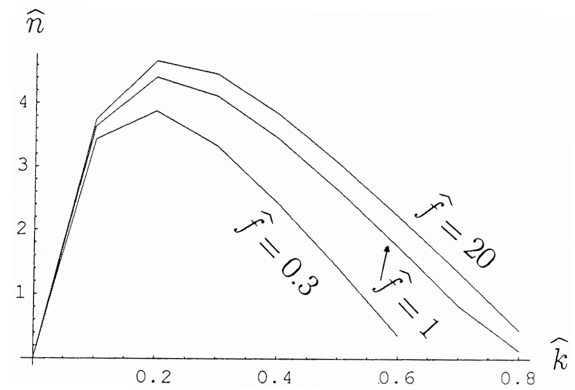

Fig. 9. The growth rate $\widehat{n}$ (multiplied by 100) plotted against the wavenumber $\widehat{k}$ for $\widehat{f} = 0.3$, 1, and 20, with $\alpha_0 = 0.7$, $\widehat{U} = 0.6$, $\widehat{\nu} = 4$, and $\widehat{\nu}_0 = 0.6$.

Note that the condition (75) is identical to the condition (65) when $\beta_1 > \beta_2$, under which we have shown a stabilizing influence of both the suspended particles and kinematic viscosity on the configuration. Thus we find that the growth rate $(\mathrm{d}n_0/\mathrm{d}\nu_0)$ and $(\mathrm{d}n_0/\mathrm{d}\nu)$ of unstable modes of the considered system are reduced with the increase of the finite Larmor radius (ν_0), and kinematic viscosity (ν), respectively, under the condition (75); whereas it is enhanced with the increase of finite Larmor radius and kinematic viscosity, respectively, if condition (76) is satisfied. In other words, the conditions (75, 76) define regions where both the finite ion Larmor radius and the kinematic viscosity have stabilizing or destabilizing influence on the growth rate of the unstable mode. It is also seen from conditions (75, 76) that they determining the regions involve the parameters f and α_0 of the suspended particles, finite Larmor radius ν_0, and kinematic viscosity ν. Note also, from equations (70–75), that when $\alpha_0 \geq 2$, then both the suspended particles and kinematic viscosity have stabilizing effects on the considered system.

The dispersion relation (69) can be written in a dimensionless form in the same manner as equation (57), and then by solving it numerically for positive real roots for various values of the parameters $\widehat{\nu}$, $\widehat{\nu}_0$, and $\widehat{f}$, respectively. Figure 7 shows the stabilizing influence of the fluid viscosity in the wavenumber range $0 \leq \widehat{k} \leq 1.2$ (approximately). Figure 8 shows also that the finite Larmor radius $\widehat{\nu}_0$ has a stabilizing effect on the considered system, and that the domain of instability is reduced by increasing $\widehat{\nu}_0$. In Figure 9, we notice that the relaxation frequency of the suspended particles $\widehat{f}$ has a destabilizing influence on the considered system composed of a uniform sheet formed by the two superposed fluids.

To emphasize the role of finite Larmor radius on the Kelvin-Helmholtz mode of two superposed hydromagnetic fluids in the absence of suspended particles. It is known that a magnetic field acting in the direction of streaming has, in general, a stabilizing effect on the Kelvin-Helmholtz configuration. A condition for stability of such a model considering superposed fluids of equal densities has been derived by Chandrasekhar [1], Shivamoggi [59] among others, is given by

$$V \geq \frac{U}{2} \tag{77}$$

where V represents Alfvén speed. It was also pointed out by Chandrasekhar [1] that a uniform magnetic field transverse to the direction of streaming does not contribute to the development of Kelvin-Helmholtz instability. We have incorporated the finite Larmor radius and kinematic viscosity corrections in the Kelvin-Helmholtz model considering a uniform magnetic field acting transverse to the direction of streaming. The criterion for stability of the medium is given by equation (65) (when $U_1 = U$, $U_2 = 0$, $\nu = \nu_0$), *i.e.*

$$k\nu_0 \geq \frac{U}{2}\cdot \tag{78}$$

On comparing equations (77, 78), we note that the quantity $k\nu_0$ in equation (78) replaced by the Alfvén speed in equation (77). This indicates the finite Larmor radius effect which arises due to the presence of the transverse magnetic field stabilizes the Kelvin-Helmholtz system in a similar manner as done by a magnetic field parallel to the direction of streaming.

5.4 Implifications of suspended particles on the medium

In this subsection, we carry out the analysis with vanishing small finite Larmor radius ($\nu_0 = 0$) for investigating implifications of suspended particles in a more complete manner. The dispersion relation (69) in this case of zero Larmor radius ($\nu_0 = 0$, and when $\alpha_1 = \alpha_2 = \alpha_0$, $\beta_1 = \beta_2 = 1/2$, $U_1 = U$, and $U_2 = -U$) reduces to

$$n^4 + [2\nu k^2 + f(2+\alpha_0)]n^3 + [4\nu k^2 f + f^2(1+\alpha_0)]n^2 + [2\nu k^2(f^2 + k^2U^2) + k^2 f U^2(\alpha_0 - 2)]n - [f^2k^2U^2(1+\alpha_0) + k^4U^4] = 0. \quad (79)$$

Clearly the absolute term of the above equation is always negative, which means that it admits at least one real positive root. However, it is interesting to evaluate the derivative of the growth rate of the unstable mode (n_0) of propagation with increasing relaxation frequency of the suspended particles as well as the kinematic viscosity. From equation (79), we find

$$\frac{\mathrm{d}n_0}{\mathrm{d}f} = -[n_0^3(2+\alpha_0) + 2n_0^2\{2\nu k^2 + f(1+\alpha_0)\} + n_0\{4\nu k^2 f - k^2U^2(2-\alpha_0)\} - 2fk^2U^2(1+\alpha_0)][4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha_0)\} + 2n_0 f\{4\nu k^2 + f(1+\alpha_0)\} + \{2\nu k^2(f^2+k^2U^2) - fk^2U^2(2-\alpha_0)\}]^{-1} \quad (80)$$

and

$$\frac{\mathrm{d}n_0}{\mathrm{d}\nu} = -2k^2 n_0[(n_0+f)^2 + k^2U^2][4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha_0)\} + 2n_0 f\{4\nu k^2 + f(1+\alpha_0)\} + \{2\nu k^2(f^2 + k^2U^2) - fk^2U^2(2-\alpha_0)\}]^{-1}. \quad (81)$$

In writing equation (79) we have considered the fact that $\alpha_0 (= mN/\rho) \leq 1$. Let us now consider the inequalities

$$[n_0^3(2+\alpha_0) + 2n_0^2\{2\nu k^2 + f(1+\alpha_0)\} + 4\nu n_0 k^2 f] \gtrless k^2U^2[n_0(2-\alpha_0) + 2f(1+\alpha_0)] \quad (82)$$

and

$$[4n_0^3 + 3n_0^2\{2\nu k^2 + f(2+\alpha_0)\} + 2n_0 f\{4\nu k^2 + f(1+\alpha_0)\} + \{2\nu k^2(f^2 + k^2U^2)\}] \gtrless fk^2U^2(2-\alpha_0). \quad (83)$$

If both upper and lower signs of the inequalities (82, 83) are simultaneously satisfied, we find that $\mathrm{d}n_0/\mathrm{d}f$ is negative, and if the upper and lower signs of equations (82, 83) or *vice versa* simultaneously hold, then $\mathrm{d}n_0/\mathrm{d}f$ turns out to be positive. From the above analysis, we find that in the absence of finite Larmor radius, the suspended particles reduces as well as increases the growth rate of the considered Kelvin-Helmholtz system. Also, we conclude from equation (81). that the growth rate of the unstable Kelvin-Helmholtz modes in this case is decreased or increased with increasing fluid viscosity if the denominator in equation (81) is positive or negative, respectively; *i.e.* if the upper or lower signs of the inequality (83) holds, respectively. This means, under the restriction (83), that the fluid viscosity has a stabilizing as well as a destabilizing effect on the considered configuration. Note also that when $\alpha_0 \geq 2$, then equation (81) shows that the kinematic viscosity has a stabilizing effect on the considered system in this case. In the absence of both viscosity and fluid velocities, a similar conclusion regarding the effect of suspended particles has been given by Chhajlani *et al.* [60] in the context of Rayleigh-Taylor instability of a stratified plasma in the presence of a uniform horizontal magnetic field.

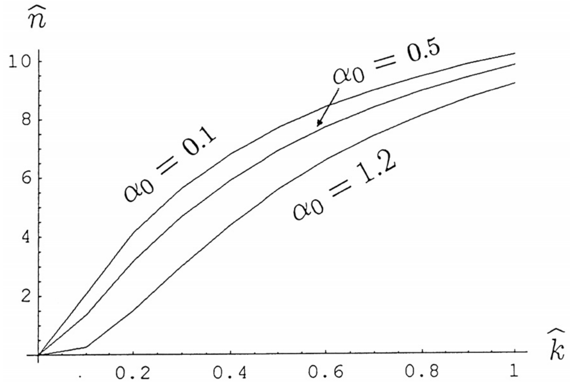

Fig. 10. The growth rate $\hat{n}$ (multiplied by 100) plotted against the wavenumber $\hat{k}$ for $\alpha_0 = 0.1$, 0.5, and 1.2, with $\hat{f} = 0.1$, $\hat{U} = 0.6$, and $\hat{\nu} = 0.8$.

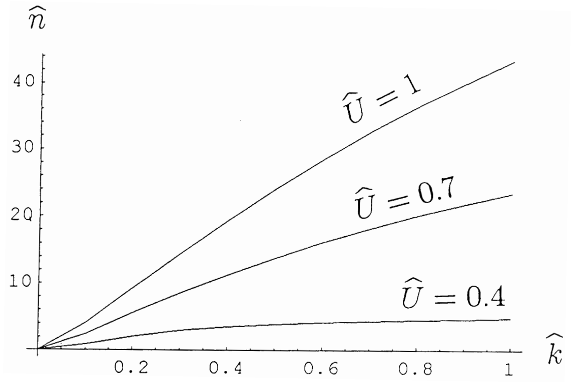

Fig. 11. The growth rate $\hat{n}$ (multiplied by 100) plotted against the wavenumber $\hat{k}$ for $\hat{U} = 0.4$, 0.7, and 1, with $\hat{f} = 0.1$, $\hat{\alpha} = 0.3$, and $\hat{\nu} = 0.5$.

Now, after writing equation (79) in a non-dimensional form in the same manner as equation (57), and then by solving it numerically for positive real roots of the growth rate $\hat{n}$, we can draw the positive real parts of $\hat{n}$ against the wavenumber $\hat{k}$ for various values of the parameters α_0 and $\hat{U}$, respectively, in the absence of finite Larmor radius effect (*i.e.* when $\hat{\nu}_0 = 0$). Figures 10 and 11 show, respectively, that the density of the suspended particles α_0 has usually a stabilizing influence on the considered system in this case, while the fluid velocity $\widehat{U}$ has a destabilizing effect, for all wavenumber values, on the considered system in the absence of the finite Larmor radius $\widehat{\nu}_0$.

6 Concluding remarks

The problem of Kelvin-Helmholtz instability, considering a uniform magnetic field along and transverse to the direction of streaming has been treated by Chandrasekhar [1]. He has noted that these two directions are profoundly different with respect to the development of Kelvin-Helmholtz instability. The magnetic field in the direction of streaming has a stabilizing influence on the Kelvin-Helmholtz instability, whereas it does not affect the stability in the transverse direction. In this regard, we have made a linear analysis of the Kelvin-Helmholtz instability of two superposed infinitely conducting and viscous fluids consisting of a uniform mixture of a gas and suspended particles in the presence of a uniform magnetic field transverse to the direction of streaming, including finite ion Larmor radius. The stability analysis is discussed analytically, and confirmed numerically, and the obtained results can be summarized as follows.

(1) In the absence of both suspended particles and finite Larmor radius, it is found that the stable configuration can be stabilized or destabilized under certain condition, and the kinematic viscosity has a stabilizing effect in this case, while the unstable configuration remains always unstable even in the presence of kinematic viscosity, and interchange perturbations remains unaffected by the presence of the magnetic field. The limiting cases of absence of gravitational field or fluid velocities are investigated too. The system is found to be unstable in the absence of gravitational field, while the stable and unstable cases remains stable and unstable, respectively, in the absence of fluid velocities. In the later case, it is found also, for ideal fluids, that the system is stable or unstable according as $\beta_1 \gtrless \beta_2$, respectively.

(2) In the absence of both fluid velocities and finite Larmor radius, it is found that both relaxation frequency of the suspended particles, and kinematic viscosity have stabilizing as well as destabilizing influences on the considered system under certain conditions, It is found also, as the size of the particles increases, that the growth rate of unstable Rayleigh-Taylor modes decreases even in the presence of fluid viscosity. The corresponding results in the absence of suspended particles have been recently obtained by Hoshoudy and El-Ansary [61].

(3) In the absence of suspended particles, and assuming that the streaming fluids with different velocities have equal densities, it is found, when the lower fluid is streaming faster than the upper one, that the system is stable or unstable under certain condition, while when the lower fluid is streaming slower than the upper one, the system is found to be usually unstable. For a vortex sheet, we observed that the finite Larmor radius has a stabilizing influence as it reduces the frequency of oscillation of the system.

(4) In the case of combined effect of finite Larmor radius, viscosity, and suspended particles, when identical gas particles composite medium (α_0) occupy the two regions, it is found, for a vortex sheet of equal densities, that both finite Larmor radius, and fluid viscosity have stabilizing as well as destabilizing effects under certain condition, and that they usually have stabilizing effects when $\alpha_0 \geq 2$. We have observed also, in the absence of suspended particles, that the finite Larmor radius (when $U_1 = U$, $U_2 = 0$, and $\nu = \nu_0$) has a stabilizing effect under the condition $k\nu_0 \geq U/2$.

(5) Considering the last case in the limit of zero finite Larmor radius, it is found that both the relaxation frequency of suspended particles, and the kinematic viscosity have stabilizing as well as destabilizing effects under certain conditions. Also the fluid viscosity is found to has a stabilizing influence when $\alpha_0 \geq 2$. The case of absence of both fluid viscosity and streaming yields the known previous results in the context.

(6) Finally, we have noted that the finite Larmor radius in the case of transverse Kelvin-Helmholtz instability plays a similar stabilizing role as the magnetic field does for perturbations parallel to the direction of streaming. Thus we conclude that a magnetic field transverse to the direction of streaming does influence the Kelvin-Helmholtz instability when the finite Larmor radius corrections are included to the analysis. Therefore, we assert that the finite Larmor radius stabilization for interchange perturbations $(\mathbf{k} \perp \mathbf{H})$ is similar to the stabilization due to a magnetic field for non-interchange $(\mathbf{k} \parallel \mathbf{H})$ perturbations.

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